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On the Approximation by Polynomials in $C^q_{[0,1]}$

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1. INTRODUCTION

Let $f: [a,b] \to R$ and let $(f_n)_n$ be a sequence of real polynomials, uniformly convergent to f on $[a,b]$.

In general, papers concerning such sequences $(f_n)_n$ can be divided into two classes:

(1) Papers concerning the conservation by the functions f_n of the properties of f (e.g., monotonicity and convexity); see [6, 7, 9, 10, 11, 12].

(2) Papers concerning the monotonicity of the sequence $(f_n)_n$; see [1–5, 8].

Let q be an integer $\geqslant 0$. In [4] we have proved the result: "if $f \in C^q_{[0,1]}$, then there exist polynomial sequences $(P_n)_n$, $(Q_n)_n$ such that for $i = 0, 1,..., q$, we have $P_n^{(i)} \to f^{(i)}$, $Q_n^{(i)} \to f^{(i)}$ uniformly, $(P_n^{(i)})_n$ being monotonically decreasing and $(Q_n^{(i)})_n$ monotonically increasing on $[0, 1]$."

In the present paper, we obtain the result of [4] in a constructive way, in the particular case $f \in C^{q+2}_{[0,1]}$, using the modified Bernstein polynomials of [1].

The construction of the sequences $(P_n)_n$, $(Q_n)_n$ in the general case is still an open question.

2. BASIC RESULT

Let $C^{q+2}_{[0,1]}$, $q \geqslant 0$, the set of $q+2$ times continuously differentiable functions on the interval $[0, 1]$, let $f \in C^{q+2}_{[0,1]}$, and let us denote $M_{q+2} = \max\{|f^{(q+2)}(x)|;\ x \in [0,1]\}$, $C_{q,i} = M_{q+2}(\frac{1}{4} + (q-i)/2)/((q+2)! - 1)$, $i = \overline{0, q}$, and

$$A_{q,m}(f;x) = f(0) + x \cdot f'(0) + \cdots + x^{q-1} \cdot f^{(q-1)}(0)/(q-1)! + \int_0^x ((x-s)^{q-1} \cdot B_m(f^{(q)};s)/(q-1)!)\,ds$$

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(where $B_m(\cdot\,; s)$ is the Bernstein polynomial). The $A_{q,m}(f; x)$ are the modified Bernstein polynomials of [1].

THEOREM 2.1. *If* $L_m(f; x) = A_{q,m}(f; x) + (1/m) \cdot \sum_{i=0}^{q} C_{q,i} \cdot x^i/i!$, $m = 1, 2,\ldots,$ *then, for* $j = 0, 1,\ldots, q$, $L_m^{(j)}(f; x) \to f^{(j)}(x)$ *uniformly and monotonically decreasing on* $[0, 1]$.

Proof. Evidently $L_m^{(j)}(f; x) \to f^{(j)}(x)$ uniformly on $[0, 1]$, for each $j = 0, 1,\ldots, q$, because $A_{q,m}^{(j)}(f; x) \to f^{(j)}(x)$ uniformly on $[0, 1]$, $\forall j = \overline{0, q}$ (see [1]). Also, in [1] it is proved that, if $f \in C^q_{[0,1]}$, then $\exists u_i \in (0, 1)$ (distinct points, depending on f, q, and j) such that

$$A_{q,m+1}^{(j)}(f; x) - A_{q,m}^{(j)}(f; x)$$
$$= -x^{q-j+1}((q-j+2)/2 - x) \cdot [u_1,\ldots, u_{q-j+3}; f^{(j)}]/(m(m+1))$$

$\forall x \in [0, 1]$, $\forall m = 1, 2,\ldots,$ and $\forall j = \overline{0, q}$, where $[u_1,\ldots, u_{q-j+3}; \cdot]$ is the divided difference of order $q - j + 2$ taken at the points $u_1,\ldots, u_{q-j+3}$. If $f \in C^{q+2}_{[0,1]}$, we get $\forall m = 1, 2,\ldots,$ and $\forall x \in [0, 1]$,

$$|A_{q,m+1}^{(j)}(f; x) - A_{q,m}^{(j)}(f; x)|$$
$$\leqslant (1/(m(m+1))) \cdot (+x^{q-j+1} \cdot ((q-j+2)/2 - x) \cdot M_{q+2})/(q+2)!$$

Let $h_j(x) = x^{q-j+1} \cdot ((q-j+2)/2 - x)$. We have

$$h_j'(x) = (q-j+1)\, x^{q-j} \cdot ((q-j+2)/2 - x) - x^{q-j+1}$$
$$= x^{q-j}(q-j+2)((q-j+1)/2 - x).$$

For $j = 0, 1,\ldots, q-1$, evidently $h_j'(x) \geqslant 0$ $\forall x \in [0, 1]$; therefore h is nondecreasing on $[0, 1]$ and $|h_j(x)| \leqslant h_j(1) = (q-j)/2$ $\forall x \in [0, 1]$. For $j = q$, we have $h_j(x) = x(1-x) \leqslant \frac{1}{4}$, $\forall x \in [0, 1]$; therefore, for each $j = \overline{0, q}$, we can write $|h_j(x)| \leqslant ((\frac{1}{4} + (q-j)/2)$ $\forall x \in [0, 1]$. Then we have

$$|A_{q,m+1}^{(j)}(f; x) - A_{q,m}^{(j)}(f; x)|$$
$$\leqslant (1/(m(m+1))) \cdot (\tfrac{1}{4} + (q-j)/2) \cdot M_{q+2}/(q+2)!$$
$$< C_{q,j}/(m(m+1)), \qquad \forall x \in [0, 1], \forall m = 1, 2,\ldots, \text{ and } \forall j = \overline{0, q}. \quad (1)$$

Let $j \in \{0, 1,\ldots, q\}$ be fixed. We have $L_m^{(j)}(f; x) = A_{q,m}^{(j)}(f; x) + (1/m) \cdot \sum_{i=0}^{q-j} C_{q,i+j} \cdot (x^i/i!)$; therefore

$$L_m^{(j)}(f; x) - L_{m+1}^{(j)}(f; x)$$
$$= A_{q,m}^{(j)}(f; x) - A_{q,m+1}^{(j)}(f; x) + C_{q,j}/(m(m+1))$$
$$+ \sum_{i=1}^{q-j} C_{q,i+j} \cdot x^i/(i!(m(m+1))) > 0, \forall x \in [0, 1],$$

and $\forall m = 1, 2, \ldots$, (from (1)). Therefore the sequence $(L_m^{(j)}(f; x))_m$ is monotonically decreasing.

Remarks. (1) Evidently, the sequence $(R_m^{(j)}(f; x))_m$, $R_m^{(j)}(f; x) = A_{q,m}^{(j)}(f; x) - (1/m) \cdot \sum_{i=0}^{q-j} C_{i+j} x^i / i!$, is monotonically increasing on $[0, 1]$, for each $j = 0, 1, \ldots, q$.

(2) For $q = 0$, we replace the polynomial $A_{q,m}(f; x)$ by the Bernstein polynomial $B_m(f; x)$. It is known that

$$B_{m+1}(f; x) - B_m(f; x) = (-x(1-x))/(m(m+1)) \cdot \sum_{i=0}^{m-1} p_{m-1,i}(x)$$
$$\cdot\ [k/m, (k+1)/(m+1), (k+1)/m; f],$$

where $p_{m-1,i}(x) = \binom{m-1}{i} \cdot x^i \cdot (1-x)^{m-1-i}$. Now, if $f \in C^2_{[0,1]}$ and $M_2 = \sup\{|f''(x)|;\ x \in [0, 1]\}$, we get $|B_{m+1}(f;\ x) - B_m(f \cdot x)| \leqslant (x(1-x))/(m(m+1))$; $M_2/2 < M_2/(4m(m+1))$. Then the polynomial $L_m(f \cdot x) = B_m(f \cdot x) + M_2/4m$ is monotonically decreasing on $[0, 1]$.

For other examples see [3].

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